1.3 Trigonometric Functions

Having set the stage by introducing the concepts of angle, triangle, similarity, proportion etc., we now turn to the basic tools of trigonometry: trigonometric functions.

Pythagorean Theorem. We simply state it for future use: In any right triangle, the following relation holds between the lengths of the legs a and b and the length of the hypothenuse c:

$$a^2 + b^2 = c^2$$

Sine and Cosine. Sine and cosine functions should be thought of as the most elementary trigonometric functions, as the other four trigonometric functions can be derived from them by purely algebraic means: once we know the value of the sine and cosine at a certain angle, we can easily calculate the its tangent, cotangent, secant and cosecant.

Definition. In order to define these functions, let us look at an angle α in standard position, and choose a point P lying on its terminal side (other than the vertex). Now, we drop a perpendicular from P to the x-axis: we thus obtained the coordinates of the point P. When we know the coordinates x and y of the point P, we can calculate its distance from the origin using the Pythagorean theorem: $r = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$.[1] Using x, y and r, we define the sine and cosine functions:

$$\sin\alpha = \frac{y}{r}$$
$$\cos\alpha = \frac{x}{r}$$

Sign. It is very important to note that x and y in the definition above are *not* the *lengths* of the sides of a triangle, but the *coordinates* of the point P, which means that they are not necessarily positive. For example, if P projects onto the negative x-axis, the value of x is negative. As r is positive, we see that the sine function at the angle α has the same sign as the y-coordinate of the point P, and that the cosine function at that angle has the same sign as the x-coordinate of the point P. The signs of sine and cosine therefore depend on the quadrant in which the angle lies.

Independence from the point P. Although we specified some point P on the terminal side of the angle, and used its coordinates in the definition of the sine and cosine functions, the choice of the point P does not affect the value of the trig functions. This can be verified by choosing another point P' on the terminal side of the same angle, and observing that its coordinates are proportional to those of P (similar triangles!), and therefore the ratios in the definition remain unchanged.

[1] Note that $r > 0$: it is different than zero because we assumed that P is not at the vertex, and it is positive because the square root on the right hand side is positive.

Tangent. The tangent function is defined as the following ratio:

$$\tan\alpha = \frac{y}{x} \quad (x \neq 0)$$

Note that, unlike the sine and cosine functions, which had the strictly positive r in the denominator, there are angles for which the tangent function is undefined: angles for which $x = 0$, i.e. angles whose terminal side lies along the (either positive or negative) y-axis.

As we indicated above, we can calculate the value of the tangent if we know the values of sine and cosine:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \quad (\cos\alpha \neq 0)$$

This is easily verified by substituting the definition for sine and cosine, and cancelling out the $1/r$.

A note on alternative definitions. As the last relation is entirely equivalent to the tangent definition above, we can use it as an alternative definition. In fact, some may find it easier to remember the definitions of trig functions in such a way as to remember the definitions of sine and cosine in terms of x, y and r, and then to remember the definitions of all other trig functions in terms of sine and cosine (remembering them all in terms of x, y and r might be more difficult, as it involves a lot of different combinations of these three values).

Secant and Cosecant. The next two trigonometric functions are secant and cosecant. They are defined by the following formulas:

$$\sec\alpha = \frac{r}{x} \quad (x \neq 0)$$
$$\csc\alpha = \frac{r}{y} \quad (y \neq 0)$$

Just as tangent, these two functions are undefined for certain angles (which ones?).

By comparing these two definitions with the ones for sine and cosine, the following relation between the two pairs of functions can be easily checked:

$$\sec\alpha = \frac{1}{\cos\alpha} \quad (\cos\alpha \neq 0)$$
$$\csc\alpha = \frac{1}{\sin\alpha} \quad (\sin\alpha \neq 0)$$

This could be perceived as an alternative definition of secant and cosecant.

Cotangent. Finally, the last trigonometric function we will encounter — cotangent — is defined by the following ratio:

$$\cot\alpha = \frac{x}{y} \quad (y \neq 0)$$

This function is undefined for angles which have their terminal sides along the (positive or negative) x-axis.

Similarly as for tangent, we can relate the cotangent function to sine and cosine:

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} \quad (\sin\alpha \neq 0)$$

which is an alternative definition for the cotangent function.